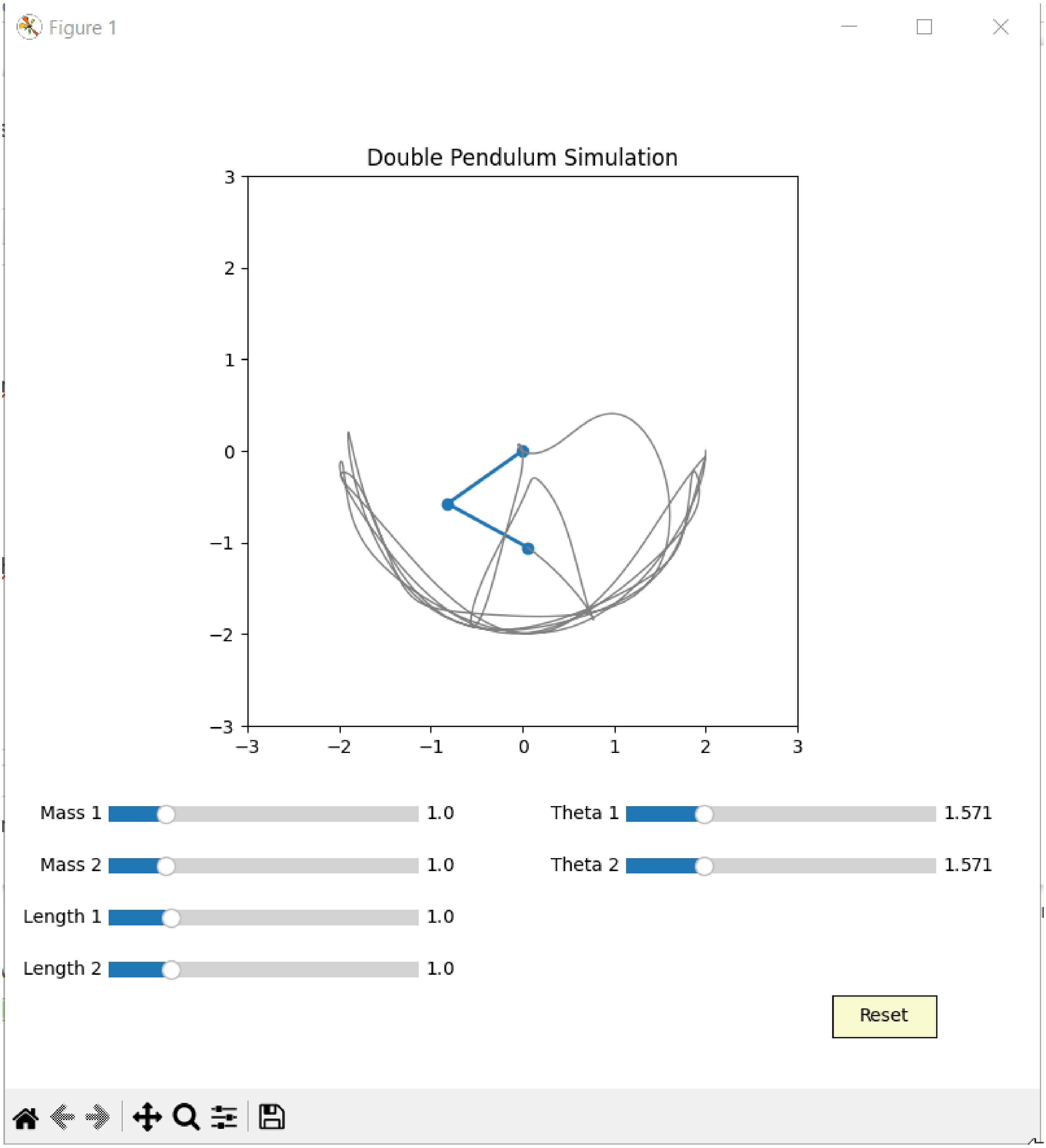
Python: Symulator wahadła podwójnego

PY: Double Pendulum Simulator



This is no ordinary pendulum - it is a **double pendulum** (_double pendulum_), one of the simplest physical systems demonstrating the phenomenon of **deterministic chaos**. Two arms, two masses, gravity - and the result is a beautiful, unpredictable dance in which a slight change in starting angle can completely alter the future motion.

What is actually going on here?

The system consists of two point masses suspended from inextensible rods. The motion is described by four variables:

- \(\theta_1 \) - the angle of the first arm relative to the vertical,
- \(\theta_2 \) – the angle of the second arm relative to the vertical,
- \(\omega_1 = \dot{\theta}_1 \) – angular velocity of the first arm,
- \(\omega_2 = \dot{\theta}_2 \) – angular velocity of the second arm.

The equations of motion are either derived from Newton's principles of dynamics or directly from Lagrange mechanics:

$$ \begin{align*} \delta &= \theta_2 - \theta_1 \\ \ddot{\theta}_1 &= \frac{m_2 l_1 \omega_1^2 \sin\delta \cos\delta + m_2 g \sin\theta_2 \cos\delta + m_2 l_2 \omega_2^2 \sin\delta - (m_1 + m_2) g \sin\theta_1}{(m_1 + m_2) l_1 - m_2 l_1 \cos^2\delta} \\ \ddot{\theta}_2 &= \frac{-(m_1 + m_2) l_1 \omega_1^2 \sin\delta + (m_1 + m_2) g \sin\theta_1 \cos\delta - m_2 l_2 \omega_2^2 \sin\delta \cos\delta - (m_1 + m_2) g \sin\theta_2}{\left(\frac{l_2}{l_1} \right)((m_1 + m_2) l_1 - m_2 l_1 \cos^2\delta)} \end{align*} $$ Does it look wild? That's what non-linear physics looks like :) The motion is simulated numerically (using the Runge-Kutta method) and animated with the help of the library **matplotlib** in Python. The code allows you to change interactively: - the masses of the two weights, - the lengths of the arms, - initial angles. Each click is a strange new world of trajectories. ===== Mathematical background: Runge-Kutta and Lagrange mechanics ===== Before a computer can simulate anything, we need two things: **a physical model** (i.e. the equations) and **a method for solving them**. For our double pendulum: - the model describes **Lagrange mechanics**, - the solution is given by **Runge-Kutta method of the 4th order (RK4)**. ---- === Lagrange mechanics === Instead of the classical forces from Newton's 2nd principle, Lagrange uses the principles of energy. We introduce the function **Lagrangian**: $$ L = T - V $$ Where: - T - kinetic energy, - V - potential energy.

For a system in generalised coordinates q_i, the equations of motion are derived from the so-called 'Lagrange equations'. **Lagrange equations**:

$$ \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0 $$

In our case: $q_1 = \theta_1$, $q_2 = \theta_2$ — arms angle.

The energies of the system are expressed as follows:

- Kinetic energy:

$$ T = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 \left[l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \right] $$

- Potential energy (in a gravitational field):

$$ V = - (m_1 + m_2) g l_1 \cos\theta_1 - m_2 g l_2 \cos\theta_2 $$

We substitute into $L = T - V$ and then insert into the Lagrange equations. The result is two second-order non-linear differential equations, which we then transform to a system of first-order equations.

The 4th order Rungego-Kutta method (RK4)

The RK4 method allows a system of first-order differential equations to be solved numerically:

$$ \dot{y} = f(t, y), \quad y(t_0) = y_0 $$

To find y_{n+1} at the point $t_{n+1} = t_n + h$, we compute:

$$ \begin{aligned} k_1 &= f(t_n, y_n) \\ k_2 &= f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_1\right) \\ k_3 &= f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_2\right) \\ k_4 &= f(t_n + h, y_n + h k_3) \\ y_{n+1} &= y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \end{aligned} $$

This is an averaging of the four state change estimates. This gives a local accuracy of the order of $\mathcal{O}(h^5)$ and a global accuracy of the order of $\mathcal{O}(h^4)$.

Our code uses `solve_ivp()` from the `scipy` library, which by default uses a version of **RK45** - adaptive method with error control and dynamic step selection.

Why it works

The system we are modelling: - is **non-linear** (angles and their derivatives appear in trigonometric functions), - exhibits **chaotic behaviour** (enormous sensitivity to initial conditions), - **has no analytical solution** - only numerical simulation allows it to be analysed.

Lagrange mechanics allows the construction of a physically correct model based on the principles of behaviour, and the Runge-Kutta method allows this model to be **effectively and accurately simulate** on a computer.

Links for the curious

- Lagrange mechanics (EN)
- Wikipedia: Deterministic chaos
- Wikipedia: Double Pendulum
- Interactive online model (MyPhysicsLab)
- MyPhysicsLab - online physical simulations
- Wolfram Demonstrations: Classical Mechanics
- Lecture notes: Numerical Analysis, Oxford

Programme code

double_pendulum.py

```
import numpy as np
import matplotlib.pyplot as plt
from matplotlib.animation import FuncAnimation
from matplotlib.widgets import Slider, Button
from scipy.integrate import solve_ivp

# Constants and default parameters
g = 9.81

# Default parameters for masses, lengths and initial angles (in
radians)
default_params = {
    'm1': 1.0,
    'm2': 1.0,
    'l1': 1.0,
    'l2': 1.0,
    'theta1': np.pi / 2,
    'theta2': np.pi / 2,
    'omega1': 0.0,
    'omega2': 0.0,
}

def double_pendulum_derivs(t, y, m1, m2, l1, l2):
    """Returns the derivatives for the double pendulum system.

    y = [theta1, theta2, omega1, omega2]
    """
    theta1, theta2, omega1, omega2 = y

    delta = theta2 - theta1

    denom1 = (m1 + m2) * l1 - m2 * l1 * np.cos(delta)**2
    domega1_dt = (m2 * l1 * omega1**2 * np.sin(delta) * np.cos(delta) +
                  m2 * g * np.sin(theta2) * np.cos(delta) +
                  m2 * l2 * omega2**2 * np.sin(delta) -
                  (m1 + m2) * g * np.sin(theta1)) / denom1

    denom2 = (l2 / l1) * denom1
    domega2_dt = (- m2 * l2 * omega2**2 * np.sin(delta) * np.cos(delta)
+
                  (m1 + m2) * g * np.sin(theta1) * np.cos(delta) -
                  (m1 + m2) * l1 * omega1**2 * np.sin(delta) -
                  (m1 + m2) * g * np.sin(theta2)) / denom2

    return [omega1, omega2, domega1_dt, domega2_dt]
```

```
def simulate(params, t_max=20, dt=0.02):
    """Simulate the double pendulum motion with given parameters."""
    t_span = (0, t_max)
    t_eval = np.arange(0, t_max, dt)
    y0 = [params['theta1'], params['theta2'], params['omega1'],
params['omega2']]
    sol = solve_ivp(double_pendulum_derivs, t_span, y0,
args=(params['m1'], params['m2'], params['l1'], params['l2']),
                    t_eval=t_eval, method='RK45')
    return sol.t, sol.y

# Initial simulation data
t, y = simulate(default_params)
theta1_vals = y[0]
theta2_vals = y[1]

def get_positions(theta1, theta2, l1, l2):
    """Calculate positions of pendulum bobs."""
    x1 = l1 * np.sin(theta1)
    y1 = -l1 * np.cos(theta1)
    x2 = x1 + l2 * np.sin(theta2)
    y2 = y1 - l2 * np.cos(theta2)
    return x1, y1, x2, y2

# Create the figure and the animation axes
fig, ax = plt.subplots(figsize=(8, 8))
plt.subplots_adjust(left=0.1, bottom=0.35)
ax.set_xlim(-3, 3)
ax.set_ylim(-3, 3)
ax.set_aspect('equal')
ax.set_title('Double Pendulum Simulation')

# Initialize line and bob markers
line, = ax.plot([], [], 'o-', lw=2)
trace, = ax.plot([], [], '-', lw=1, color='gray')  # Optional trace of
second bob
trace_x, trace_y = [], []

def init():
    line.set_data([], [])
    trace.set_data([], [])
    return line, trace

# Animation update function
def update(frame):
    theta1 = theta1_vals[frame]
    theta2 = theta2_vals[frame]
    x1, y1, x2, y2 = get_positions(theta1, theta2,
current_params['l1'], current_params['l2'])
    line.set_data([0, x1, x2], [0, y1, y2])
    trace_x.append(x2)
```

```
    trace_y.append(y2)
    trace.set_data(trace_x, trace_y)
    return line, trace

# Create sliders for initial parameters
axcolor = 'lightgoldenrodyellow'
ax_m1 = plt.axes([0.1, 0.25, 0.3, 0.03], facecolor=axcolor)
ax_m2 = plt.axes([0.1, 0.20, 0.3, 0.03], facecolor=axcolor)
ax_l1 = plt.axes([0.1, 0.15, 0.3, 0.03], facecolor=axcolor)
ax_l2 = plt.axes([0.1, 0.10, 0.3, 0.03], facecolor=axcolor)
ax_theta1 = plt.axes([0.6, 0.25, 0.3, 0.03], facecolor=axcolor)
ax_theta2 = plt.axes([0.6, 0.20, 0.3, 0.03], facecolor=axcolor)

slider_m1 = Slider(ax_m1, 'Mass 1', 0.1, 5.0,
valinit=default_params['m1'])
slider_m2 = Slider(ax_m2, 'Mass 2', 0.1, 5.0,
valinit=default_params['m2'])
slider_l1 = Slider(ax_l1, 'Length 1', 0.5, 3.0,
valinit=default_params['l1'])
slider_l2 = Slider(ax_l2, 'Length 2', 0.5, 3.0,
valinit=default_params['l2'])
slider_theta1 = Slider(ax_theta1, 'Theta 1', 0, 2*np.pi,
valinit=default_params['theta1'])
slider_theta2 = Slider(ax_theta2, 'Theta 2', 0, 2*np.pi,
valinit=default_params['theta2'])

# Dictionary to hold current simulation parameters
current_params = default_params.copy()

def update_simulation(val):
    """Update simulation based on slider values."""
    global t, y, theta1_vals, theta2_vals, trace_x, trace_y,
current_params

    # Update current parameters from sliders
    current_params['m1'] = slider_m1.val
    current_params['m2'] = slider_m2.val
    current_params['l1'] = slider_l1.val
    current_params['l2'] = slider_l2.val
    current_params['theta1'] = slider_theta1.val
    current_params['theta2'] = slider_theta2.val
    current_params['omega1'] = 0.0
    current_params['omega2'] = 0.0

    # Re-run the simulation with new parameters
    t, y = simulate(current_params)
    theta1_vals = y[0]
    theta2_vals = y[1]

    # Clear the trace and reset animation frame index
    trace_x.clear()
```

```
    trace_y.clear()
    ani.frame_seq = ani.new_frame_seq()
    fig.canvas.draw_idle()

# Call update_simulation when any slider value changes
slider_m1.on_changed(update_simulation)
slider_m2.on_changed(update_simulation)
slider_l1.on_changed(update_simulation)
slider_l2.on_changed(update_simulation)
slider_theta1.on_changed(update_simulation)
slider_theta2.on_changed(update_simulation)

# Button to reset sliders to default values
reset_ax = plt.axes([0.8, 0.05, 0.1, 0.04])
button_reset = Button(reset_ax, 'Reset', color=axcolor,
hovercolor='0.975')

def reset(event):
    slider_m1.reset()
    slider_m2.reset()
    slider_l1.reset()
    slider_l2.reset()
    slider_theta1.reset()
    slider_theta2.reset()

button_reset.on_clicked(reset)

# Create the animation
ani = FuncAnimation(fig, update, frames=len(t), init_func=init,
interval=20, blit=True)

plt.show()
```

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